The Minkowski Problem for hedgehogs: uniqueness results

Y. Martinez-Maure

Institut Mathématique de Jussieu, Paris

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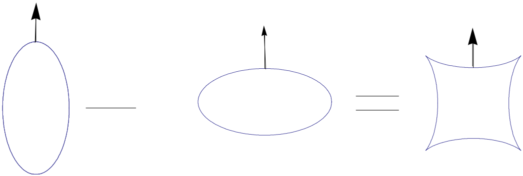
The CMP admits a natural extension to hedgehogs

- **The classical Minkowski problem (CMP)**:

Existence, uniqueness and regularity of a closed convex hypersurface of $\mathbb{R}^{n+1}$ whose Gauss curvature is prescribed as a positive function on $\mathbb{S}^n$.

- **Central role in:**
  - the theory of convex bodies.
  - the theory of elliptic Monge-Ampère equations.
- **The CMP admits a natural extension to hedgehogs.**



Hedgehogs = Minkowski differences of convex bodies (or hypersurfaces)

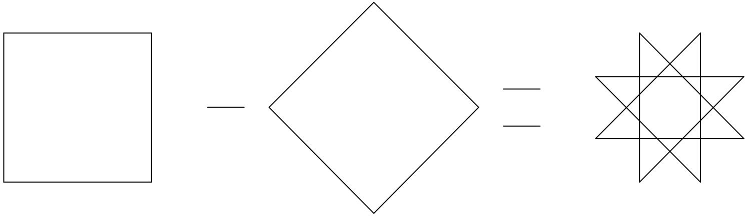
- **A way for exploring Monge-Ampère equations of mixed type.**

Hedgehogs as differences of convex bodies

- Let $\left(\mathcal{K}^{n+1}, +, .\right)$ be the set of convex bodies of $\mathbb{R}^{n+1}$ equipped with Minkowski addition and multiplication by nonnegative real numbers:

  $\mathcal{K} + \mathcal{L} = \{x + y \,|x \in \mathcal{K}, y \in \mathcal{L}\}$;

  $\lambda.\mathcal{K} = \{\lambda x \,|x \in \mathcal{K}\}$.
- $\left(\mathcal{K}^{n+1},+, .\right)$ is not a linear space: no subtraction in $\mathcal{K}^{n+1}$.
- Formal differences of convex bodies of $\mathbb{R}^{n+1}$ do constitute a linear space $\left(\mathcal{H}^{n+1}, +, .\right)$.
- Any formal difference $\mathcal{K} - \mathcal{L}$ of two convex bodies $\mathcal{K}$, $\mathcal{L} \in \mathcal{K}^{n+1}$ has a nice geometrical representation in $\mathbb{R}^{n+1}$, (Y.M.^{2}, Canad. J. Math 2006).



Case of convex bodies with positive Gauss curvature

- Subtracting two convex hypersurfaces (with positive Gauss curvature) by subtracting the points corresponding to a same outer unit normal to obtain a (possibly singular and self-intersecting) hypersurface:

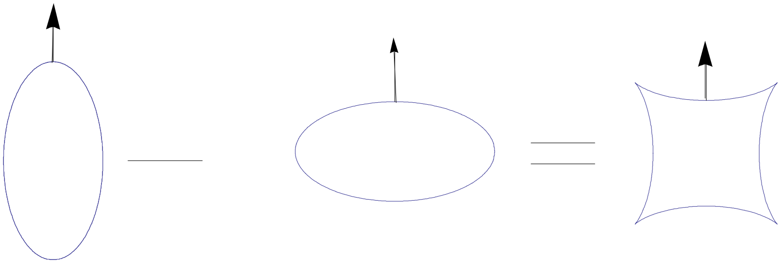


Figure: Hedgehogs as differences of convex bodies of class C^2_+

Interest of hedgehogs

- To study convex bodies or hypersurfaces by decomposition into a sum of hedgehogs.

Ex: Study of a conjectured characterization of the sphere (Y.M.[2], C. R. Acad. Sci. Paris 2001).

Idea: $S = S(0_{\mathbb{R}^3}; r) + (S - S(0_{\mathbb{R}^3}; r))$ and study of $(S - S(0_{\mathbb{R}^3}; r))$.

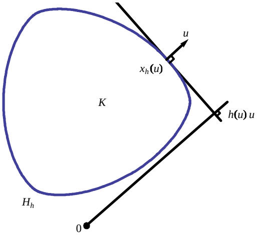
- To geometrize analytical problems by considering functions as support functions.

Ex: Geometrical proof of the Sturm-Hurwitz theorem (Y.M.[2], Arch. Math. 2003).

Support functions

Every $K \in \mathcal{K}^{n+1}$ is determined by its support function

$$\begin{aligned} h_{\mathcal{K}} \quad &: \mathbb{S}^n \longrightarrow \mathbb{R} \\ & u \longmapsto \sup\left\{\langle x, u\rangle \, x \in \mathcal{K}\right\}. \end{aligned}$$



A closed convex hypersurface of class C^2_+ is determined by its support function $h \in C^2(\mathbb{S}^n; \mathbb{R})$ as the envelope $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ of the hyperplanes $\langle x, u\rangle = h(u)$.

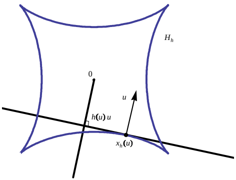
Parametrization

The natural parametrization of the envelope $\mathcal{H}_h$ of the hyperplanes with equation $\langle x, u\rangle = h(u)$ assigns to each $u \in \mathbb{S}^n$, the unique solution of the system

$$\begin{cases} \langle x, u\rangle = h(u) \\ \langle x, .\rangle = dh_u(.) \end{cases},$$

that is $x_h(u) = h(u)u + (\nabla h)(u)$. In fact, $\mathcal{H}_h = x_h(\mathbb{S}^n)$ is defined for any $h \in C^2(\mathbb{S}^n; \mathbb{R})$. It is called hedgehog with support function h.

At each regular point $x_h(u) \in \mathcal{H}_h$
u is normal to $\mathcal{H}_h$.



Gauss curvature

• The singularities of $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ are the very points where the Gauss curvature $\kappa_h(u) = 1/\det[T_p x_h]$ is infinite.

• The curvature function $R_h := 1/\kappa_h$ is well-defined and continuous on $\mathbb{S}^n$, so that **the Minkowski Problem arises for hedgehogs**.

• A calculation gives: $R_h(u) = \det[H_{ij}(u) + h(u)\delta_{ij}]$, where $(H_{ij}(u))$ is the Hessian of h at u with respect to an orthonormal frame on $\mathbb{S}^n$.

Case $n = 2$

• The curvature function of $\mathcal{H}_h \subset \mathbb{R}^3$ is given by

$$1/\kappa_h = h^2 + h\Delta_2 h + \Delta_{22} h$$

(Δ_2 is the Laplacian and Δ_{22} the Monge-Ampère operator, i.e. the sum and the product of the eigenvalues of *Hess h*).

• The type of the equation $h^2 + h\Delta_2 h + \Delta_{22} h = 1/\kappa$ is given by $sgn[1/\kappa]$. So, the PB leads to **PDE's of mixed type for non-convex hedgehogs**.

Key results on the CMP

- Major contributions by Minkowski, Alexandrov, Nirenberg, Pogorelov, Cheng-Yau and others.
- Existence of a weak solution:

Theorem (Minkowski - 1903)

If $\kappa \in C(\mathbb{S}^n; \mathbb{R})$ is positive and such that

$$\int_{\mathbb{S}^n} \frac{u}{\kappa(u)} \, d\sigma(u) = 0$$

then κ is the Gauss curvature of a unique (up to translation) closed convex hypersurface $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$.

- Strong result:

Theorem (Pogorelov - 1975, Cheng and Yau - 1976)

If $\kappa \in C^m(\mathbb{S}^n; \mathbb{R})$, with $m \geq 3$, then: $\forall \alpha \in \,]0,1[\,$, $h \in C^{m+1,\alpha}(\mathbb{S}^n; \mathbb{R})$.

Existence problem

EXISTENCE OF A C^2-SOLUTION:

What are necessary and sufficient conditions for $R \in C(\mathbb{S}^n; \mathbb{R})$ to be the curvature function of some hedgehog $\mathcal{H} = \mathcal{K} - \mathcal{L}$?

- Integral condition (1) $\int_{\mathbb{S}^n} uR(u)\, d\sigma(u) = 0$ is still necessary (but of course not sufficient: consider -1).

- Equations with no solution (Y.M.[2], Adv. in Math. 2001):

For every $v \in S^2$, $R(u) = 1 - 2\langle u, v\rangle^2$ satisfies (1) and changes sign cleanly on S^2 but is not a curvature function:

$$\text{there is no } h \in C^2(\mathbb{S}^2; \mathbb{R}) \text{ such that } R_h = R.$$

- Can the curvature function of a hedgehog $\mathcal{H}_h$ be nonpositive on $\mathbb{S}^2$? This problem is equivalent to the following conjecture:

Hedgehog with everywhere nonpositive function

Conjecture (C): *If $S \subset \mathbb{R}^3$ is a closed convex surface of class C^2_+ such that*

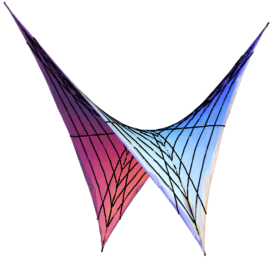
$$(k_1 - c)(k_2 - c) \leq 0,$$

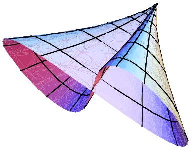
with $c = cst$, then S must be a sphere of radius $1/c$.

(C) is equivalent to (H):

(H) *If $H_h \subset \mathbb{R}^3$ is a hedgehog such that $R_h \leq 0$, then H_h is a point.*

Counter-example to (H) (Y.M.[2], C. R. Acad. Sci. Paris 2001).





Uniqueness problem

UNIQUENESS OF A C²-SOLUTION:

Let $R \in C(\mathbb{S}^n;\mathbb{R})$ be the curvature function of some hedgehog $\mathcal{H}_h$. What are necessary and sufficient conditions on R for $\mathcal{H}_h$ to be uniquely determined by R (up to parallel translations and identifying h with $-h$)*?*

In the convex case, the uniqueness comes from the equality condition in a well-known Minkowski's inequality. This inequality cannot be extended to hedgehogs and uniqueness is lost.

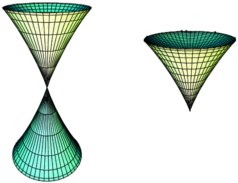


Figure: Noncongruent smooth (but not analytic) hedgehogs with the same curvature function

QUESTION. *Does there exist any pair of noncongruent analytic hedgehogs with the same curvature function?*

Results relative to the uniqueness

Let H_3 be the linear space of C^2-hedgehogs defined up to a translation in $\mathbb{R}^3$.

Theorem (Y.M.[2], Central European J. Math. 2012). *Let $\mathcal{H}$ and $\mathcal{H}'$ be C^2-hedgehogs that are linearly independent in H_3. If some linear combination of $\mathcal{H}$ and $\mathcal{H}'$ is of class C^2_+, then $\mathcal{H}$ and $\mathcal{H}'$ have distinct curvature functions.*

Our second result relies on the extension to hedgehogs of the notion of mixed curvature function.

Theorem (Y.M.[2], Central European J. Math. 2012). *Let $\mathcal{H}$ and $\mathcal{H}'$ be analytic (resp. projective C^2) hedgehogs of $\mathbb{R}^3$ that are linearly independent in H_3. If the mixed curvature function of $\mathcal{H}$ and $\mathcal{H}'$ does not change sign, then $\mathcal{H}$ and $\mathcal{H}'$ have distinct curvature functions.*

Example of a uniqueness result

The following result relies on the decomposition of hedgehogs into centered and projective parts.

Theorem (Y.M.[2], Central European J. Math. 2012). *Let $\mathcal{H}$ and $\mathcal{H}'$ be C^2-hedgehogs that are linearly independent in* H_3 *and the centered parts of which are non-trivial and proportional to one and the same convex surface of class C^2_+. Then $\mathcal{H}$ and $\mathcal{H}'$ have distinct curvature functions.*

Corollary. *Two C^2-hedgehogs of nonzero constant width that are linearly independent in* H_3 *must have distinct curvature function.*

Consequence. The Monge-Ampère equation $h^2 + h\Delta_2 h + \Delta_{22} h = R$, $R \in C\left(\mathbb{S}^2;\mathbb{R}\right)$, cannot admit more than one solution of the form $f + r$, where $f \in C^2\left(\mathbb{S}^2;\mathbb{R}\right)$ is antisymmetric and r is a nonzero constant.

(Solutions are identified if they are opposite or if they differ by the restriction to $\mathbb{S}^2$ of a linear form on $\mathbb{R}^3$)

Thank you very much

Thank you very much for your attention!

Figure: European hedgehog