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Singularities of virtual polytopes

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Abstract. A smooth hedgehog in $\mathbb{R}^3$ (that is, a parallel surface of a smooth convex body) can have singularities. It is proven by Langevin et al. that the numbers of swallowtails together with the numbers of elliptic and hyperbolic components of a generic smooth hedgehog satisfy some linear relation. In the paper, we discuss the discrete (i.e., the piecewise linear) counterpart of the same theory, define swallowtails and cuspidal edges for the discrete case, and derive an analogous formula. Here we make use of the theory of virtual polytopes that arise quite naturally in the context as the piecewise linear counterpart of smooth hedgehogs. We also present a discussion on the open problem of existence of a generic projective hedgehog without swallowtails.

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Keywords. Singularities, wave fronts, piecewise linear surfaces, hedgehogs, virtual polytopes.

1. Preliminaries

Singularities of smooth hedgehogs

Classical hedgehogs of the linear Euclidean space $\mathbb{R}^3$ are envelopes parameterized by their Gauss map. In a sense, they are geometric representations of Minkowski differences of smooth convex bodies, as presented in [4].

More precisely, the construction is as follows. Each smooth function h defined on $\mathbb{S}^2$ (or, equivalently, each positively homogeneous smooth function defined on $\mathbb{R}^3 \setminus \{O\}$) yields a family of planes $(x, \xi) = h(\xi)$. Each of the planes is cooriented by the vector ξ.

The envelope $\mathcal{H}_h$ of this family is called *the hedgehog with support function h*.

A generic hedgehog is a (possibly singular) closed surface with *elliptic* and *hyperbolic* regions. The latter are defined by the type of the intrinsic metric

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on the surface $\mathcal{H}_h$ (either elliptic or hyperbolic). Equivalently, the regions are defined by the sign of the Gaussian curvature, which is given by $K_h = \det(\nabla^2 h + h\sigma)^{-1}$, where $\nabla^2 h$ is the Hessian of h and σ the standard metric on $\mathbb{S}^2$ [6]). These regions are separated by cuspidal edges on which $\det(\nabla^2 h + h\sigma)$ is equal to 0 (or, loosely speaking, where the Gaussian curvature is infinite).

The coorientations of the planes $(x, \xi) = h(\xi)$ yield a local coorientation of the surface $\mathcal{H}_h$ at all non-singular points. Locally, the coorientation is one and the same provided we stay away from singularities, and it switches when crossing transversely a cuspidal edge.

For a generic hedgehog $\mathcal{H}_h \subset \mathbb{R}^3$, the only singularities are cuspidal edges and swallowtails which are the cusp points of cuspidal edges. This directly follows from [2]. Remind that one can distinguish two types of swallowtails (negative and positive) according to the sign of the Gaussian curvature on the tail.

A generic hedgehog satisfies the following counting formula [4].

$$\frac{x_+ - x_-}{2} = r_+ - r_- = \frac{q_+ - q_-}{2} + 1, \tag{1}$$

where x_+ (resp., x_-) is the Euler characteristic of the union of the elliptic (resp., hyperbolic) regions; r_+ (respectively, r_-) is the number of elliptic (resp., hyperbolic) regions; q_+ and q_- are the number of positive and negative swallowtails.

A hedgehog is called *projective* if its support function is odd, that is,

$$h(x) \equiv -h(-x).$$

For projective hedgehogs, pair of antipodal points on the unit sphere $\mathbb{S}^2$ correspond to a same point on the surface $\mathcal{H}_h$. This means that a projective hedgehog is a doubly covered surface. So, generic projective hedgehogs can be regarded as models of the real projective plane whose Gauss map is a bijection from the model onto the real projective plane.

Well-known are the classical models of the real projective plane in $\mathbb{R}^3$: the Boy surface, and the Steiner's Roman surface [1].

Concerning their spherical image, Hilbert and Cohn-Vossen wrote in [3]: "Unfortunately, the way in which it is distributed over the unit sphere has not yet been studied".

Let us look at the Steiner's Roman surface in more details. Classically, it has six pinch points. The support function

$$h(x, y, z) = x\left(x^2 - 3y^2\right) + 2z^3, \quad (x, y, z) \in \mathbb{S}^2,$$

gives a projective hedgehog $\mathcal{H}_h$, regarded as a hedgehog version of the Steiner's Roman surface, see Fig. 1. It has a threefold axis of symmetry and three lines of self-intersection whose endpoints are three swallowtails and three D4 singularities (so that there are six swallowtails and six D4 singularities if we consider that the surface is covered twice).

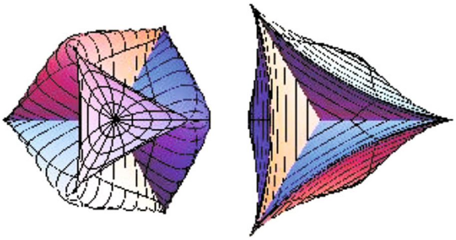

FIGURE 1 Steiner's Roman surface

One can make the singularities generic by the perturbation

$$h(x,y,z) = x(x^2 - 3y^2) + 2z^3 + 2y(y^2 - 3x^2)z^2, \quad (x,y,z) \text{ in } \mathbb{S}^2,$$

which yields a surface with six positive swallowtails (or twelve positive swallowtails for the double covered surface).

In this respect we mention the still unsolved problem raised in [4]:

Problem 1. *Does there exist a generic projective hedgehog without any swallowtail?*

Such a hedgehog would be an immersion of the projective plane into $\mathbb{R}^3$ with a bijective Gaussian mapping. Such an immersion would be smooth everywhere except for some cuspidal edges.

Discrete hedgehogs, or virtual polytopes

In the present paper, we develop a piecewise linear version of the above described theory. This means that we treat the polytopal version of smooth hedgehogs, the so-called *discrete hedgehogs*, or *virtual polytopes*, see [7] and [6]. Similarly to smooth hedgehogs, the latter appeared in the literature as geometrization of Minkowski differences of convex polytopes. In the next section we first briefly remind the reader what they are. Next, we consider some examples providing an intuition of what is the counterpart of generic singularities for virtual polytopes. Motivated by the examples, we define swallowtails and cuspidal edges in the framework of virtual polytopes and prove the discrete version of the above formula (1).

Next, we discretize the aforementioned Problem 1:

Does there exist a generic projective virtual polytope without swallowtails?

This problem remains unsolved. However, we present some discussion on the subject and give an affirmative answer to some weak version of the problem.

We wish to stress that the discrete technique already proved to be useful for problems related to smooth hedgehogs [7]. It gives a way of understanding "how a complicated smooth hedgehog can look like" by presenting a rough pattern of its shape.

In many cases the technique simplifies problems under consideration by reducing them to an analysis of some spherical drawings (rather than 3D shapes) [7,8].

It remains unclear how to smoothen a discrete hedgehog, or how to discretize a smooth hedgehog without causing extra singularities. However, it is a usual phenomenon: the interplay between smooth and discrete theories is always a difficult issue. This was the case in discrete Morse theory, in (combinatorial vs topological) characteristic classes, and in other fields.

Conventions about figures

Most of the figures depict fragments of graphs embedded in the sphere $\mathbb{S}^2$. We prefer to present the image of a spherical drawing under a central projection onto some plane. Such a projection takes geodesics to line segments.

We depict colored graphs with red and blue edges. Aiming to distinguish the colors in a black-and-white printout, we make all the red edges dashed.

2. Simplicial virtual polytopes and their fans

Simplicial virtual polytopes

In the subsection we very briefly recall definitions and constructions for virtual polytopes in $\mathbb{R}^3$, referring the reader to [7] for further reading.

A *simplicial virtual polytope* in $\mathbb{R}^3$ is a pair

$$(H, \Sigma),$$

where H is a closed simplicial surface in $\mathbb{R}^3$, and Σ is a graph embedded in the unit sphere $\mathbb{S}^2$.

We assume that for each facet of H a coorienting vector is fixed. We refer to it as to the "normal vector of the facet".

By definition, we demand that the pair (H, Σ) satisfies the conditions.

(1) The edges of the graph Σ are geodesic segments.
(2) The vertices of the graph are the endpoints of the normal vectors of facets of H. This gives a duality correspondence between vertices of Σ and the facets of H.
(3) The graph Σ is combinatorially dual to the surface H. In particular, this means that two vertices of the graph are connected by an edge if and only if the two corresponding faces of H share an edge.

The embedded graph Σ is called *the fan* of the virtual polytope.

The tiles of $\mathbb{S}^2$ generated by Σ are called *the faces of the fan*.

By duality, the vertices of the fan correspond to the facets of H, and vice versa.

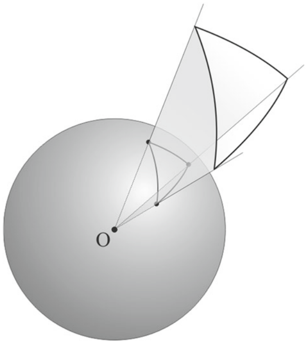

FIGURE 2 A spherical polygon yields a cone with the apex at O

For example, a simplicial convex polytope together with its outer normal fan gives such a pair.

Support function of a virtual polytope

Assume we have a simplicial virtual polytope (H, Σ). The graph Σ generates a tiling $\overline{\Sigma}$ of $\mathbb{R}^3$ into a union of cones with a common apex at the origin O (see Fig. 2).

The virtual polytope yields a positively homogeneous continuous function $h : \mathbb{R}^3 \to \mathbb{R}$ which is piecewise linear with respect to $\overline{\Sigma}$. The restriction of h on each conical tile is defined as the support function of the vertex of H which corresponds to the tile by duality.

The function h is called the *support function of the virtual polytope* (H, Σ).

Since the function h can be expressed as the (pointwise) difference of two convex functions, it makes sense to think of the pair (H, Σ) as the geometrization of the Minkowski difference of two convex polytopes.

Coloring on the fan

Observe that for a simplicial virtual polytope, the graph Σ carries some extra structure: along each of the edges of Σ the function h is either convex or concave. According to this, we color the edges either red (if h is locally convex) or blue (if h is locally concave).

An easy analysis shows that at each vertex of Σ only four types of colorings are possible (see Fig. 3).

Namely, we have the following cases.

(1) All the angles incident to a vertex are smaller than π. Then all the adjacent edges are of one and the same color. If all of them are red (respectively, blue), the function h is convex (respectively, concave) in a neighborhood of the vertex.
In both cases we color the vertex black and call it an *elliptic* vertex.

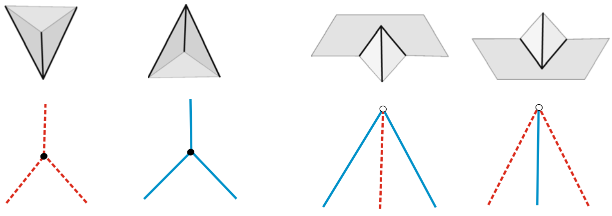

FIGURE 3 Elliptic (*1*, *2*) and hyperbolic (*2*, *3*) vertices of the fan. We depict here fragments of the fan together with the graph of the support function

(2) One of the incident angles is reflex. Then the colors of the edges can be only as is depicted in Fig. 3. That is, the edges adjacent to the reflex angle are of one and the same color, and the third edge is of the different color. In both cases we color the vertex white and call it a *hyperbolic* vertex.

The idea of the paper is that we work with the fan Σ, which is just a planar colored drawing, rather than with the (possibly) complicated 3-dimensional surface H.

The notion of genericity that we define below will be justified by a density result in Sect. 5.

Definition 1. *A simplicial virtual polytope* (H, Σ) *is generic if the following conditions hold:*

(1) *Each face of the fan* Σ *is disk-homeomorphic.*
(2) *Each face of the fan fits in an open hemisphere.*
(3) *For each face* σ *of the fan, when going along the boundary of* σ*, the color of* **vertices** *changes at most twice.*
(4) *For each face* σ *of the fan, when going along the boundary of* σ*, the color of* **edges** *changes at most four times.*

We shall need the following lemma.

Lemma 1. *Let* (H, Σ) *be a generic virtual polytope, let* σ *be a face of the fan* Σ*. Assume that all the vertices* σ *are white. Then the color of* **edges** *changes at least four times.*

Proof. Observe that in the above setting, when passing a vertex, the color of the edges does not change if and only if the angle at the vertex is greater than π. The polygon σ has at least three angles smaller than π, therefore, the edge color changes at least three times. Since the number of color changes is always even, we complete the proof. □

Now we wish to present a discrete counterpart of cuspidal edges and swallowtails for generic simplicial virtual polytopes. Firstly, we give some theoretical heuristic motivations.

For the smooth case, a “swallowtail” can be viewed as “a singular point on the surface together with some specific behavior of the surface in its neighborhood”. Consequently, for a virtual polytope (H, Σ), a “swallowtail” is expected to be “a vertex of the surface H together with some specific behavior of the surface in its neighborhood”. Since we wish to reduce the construction to the fan Σ, a “swallowtail” in this dual setting is expected to be “a face of the fan Σ together with some specific coloring on the boundary of the face”.

Analogously, for the smooth case, a “cuspidal edge” can be viewed as “one-parametric continuous family of singular points together with some specific behaviour of the surface in their neighborhoods”.

Consequently, for a virtual polytope (H, Σ), a “cuspidal edge” is expected to be “a one-parametric set of vertices of the surface H connected by edges in a cycle together with some specific behaviour of the surface H in their neighborhoods”. In the dual setting, that is, for the fan Σ, a “cuspidal edge” is expected to be a collection of faces of the fan Σ together with some specific coloring on the boundaries of the faces”. This means that one expects here a chain of “cuspidal faces” forming a “belt” of edge-to-edge adjacent faces of the fan.

We conclude that the vertices of H (or, equivalently, the faces of Σ) will be classified as elliptic, hyperbolic, swallowtails, or cuspidal ones.

Another bunch of motivating examples comes from geometry. Assuming that (H, Σ) is some virtual polytope, each of the below examples presents a vertex S of the surface H together with adjacent facets. On the right-hand side of the figures we depict the corresponding face σ of Σ and indicate the colors of vertices and edges.

We stress that the examples are not yet definitions. However, all the examples fit the corresponding definitions that are coming later in this section.

Example 1. (Fig. 4) The vertex S looks locally as a vertex of a convex polytope, the normal vectors of the incident to S facets point outwards. The face σ is a convex spherical polygon with all vertices black and all edges red.

If all the normal vectors point inwards, we have the same figure except that all the edges are blue.

Example 2. (Fig. 5) A saddle vertex S together with the corresponding face σ.

Example 3. (Fig. 6) The vertex S together with adjacent faces resembles a smooth positive swallowtail. Similarly to the smooth case, we have two lines of self-intersection which end up at the point S.

Example 4. (Fig. 7) We consider the vertex S as a discrete version of a smooth negative swallowtail. Again, we have two lines of self-intersection ending at the point S.

Example 5. (Fig. 8) The vertex S is a discrete version of a point lying on a cuspidal edge. As we mentioned in the Introduction, the coorientation of a

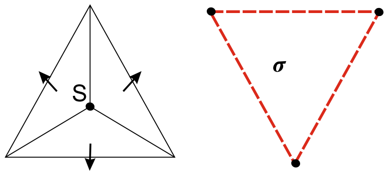

Figure 4 An elliptic vertex of H

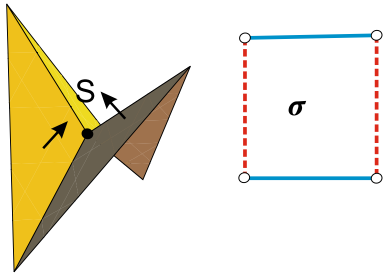

Figure 5 A saddle vertex of H

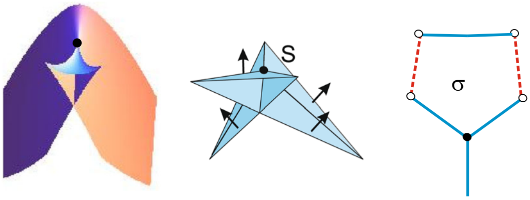

Figure 6 A smooth positive swallowtail, a discrete positive swallowtail, and the corresponding face of Σ

hedgehog changes when crossing a cuspidal edge. For this particular example the coorientation of the piecewise linear surface H also changes, so here one expects a discrete analogue of a cuspidal edge.

Motivated by the above, we define singularities for a generic virtual polytope (H, Σ).

In our setting, singularities are some of the vertices of H, or, equivalently, some of the faces of Σ.

Definition 2. *According to the genericity conditions, the faces of the fan (and, consequently, the vertices of H) fall into the following categories (see Fig. 9).*

(1) *All the vertices of the face σ are black. This implies that all the edges of σ and all incident edges are of one and the same color. We call such a face (resp., the corresponding vertex of H) an elliptic face (resp., an elliptic vertex).*
If the edge's color is red, we say that it is convex, otherwise we call it concave.

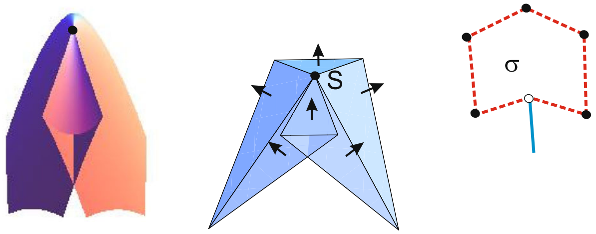

FIGURE 7 A smooth negative swallowtail, a discrete negative swallowtail, and the corresponding face of Σ

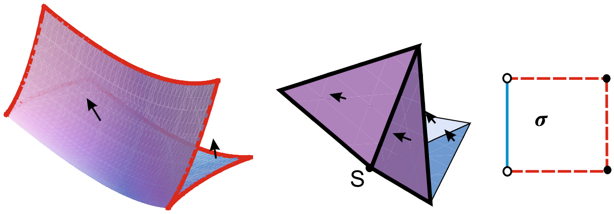

FIGURE 8 A smooth cuspidal edge and a vertex S lying on a discrete cuspidal edge

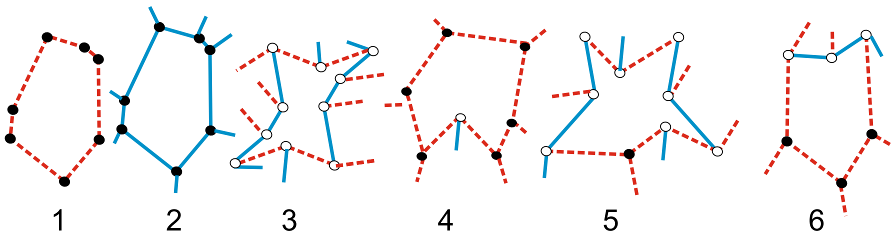

FIGURE 9 Convex elliptic face (*1*), concave elliptic face (*2*), hyperbolic face (*3*), negative swallowtail (*4*), positive swallowtail (*5*), cuspidal face (*6*)

(2) *All the vertices of the face σ are white. By Lemma 1, the color of the edges changes at least four times. The genericity condition implies that it changes exactly four times, and the face has exactly four angles smaller than π. We call such a face (resp., the corresponding vertex of H) a hyperbolic face (resp., a hyperbolic vertex).*

(3) *The vertices of the face are of different colors, and the color of the edges does not change. We call such a face (and the corresponding vertex of H) a negative swallowtail.*

(4) *The vertices of the face are of different colors, and the color of the edges changes four times. We call such a face (and the corresponding vertex of H) a positive swallowtail.*

(5) *The vertices of the face are of different colors, and the color of the edges changes exactly twice. We call such a face (resp., the corresponding vertex of H) a cuspidal face (resp., a cuspidal vertex).*

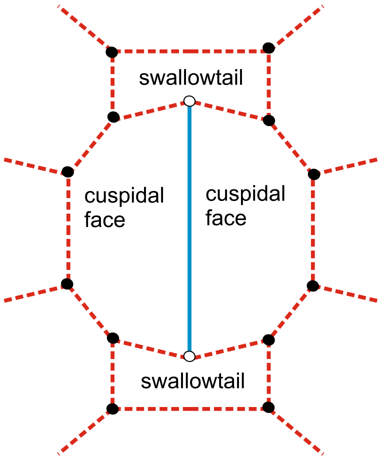

Figure 10 The fan of a virtual polytope

Definition 3. *Take the graph Σ and remove all the white vertices together with incident edges. The rest of the graph falls into disjoint union of connected components. Take one of the components C and add to it all the faces σ having all vertices in C. The result is called an elliptic region of Σ. A hyperbolic region is defined analogously.*

Example 6. Figure 10 depicts the fan of a virtual polytope. The (unique) hyperbolic region consists of two white vertices and the connecting edge. There are two negative swallowtails and two cuspidal faces. The rest is the elliptic region.

Figure 11 depicts one more fragment of a fan together with regions. Obviously, all the regions are disjoint, and they are separated by faces possessing both white and black vertices. The latter are cuspidal faces and swallowtails.

3. The relation for singularities

Theorem 1. *For a generic virtual polytope (H, Σ) we have*

$$(1/2)(X_+ - X_-) = R_+ - R_- = (Q_+ - Q_-)/2 + 1.$$

Here X_+ (resp., X_-) is the Euler characteristic of the union of elliptic (resp., hyperbolic) regions; R_+ (resp., R_-) is the number of elliptic (resp., hyperbolic) regions; Q_+ and Q_- are the numbers of positive and negative swallowtails.

The proof is based on the following lemma.

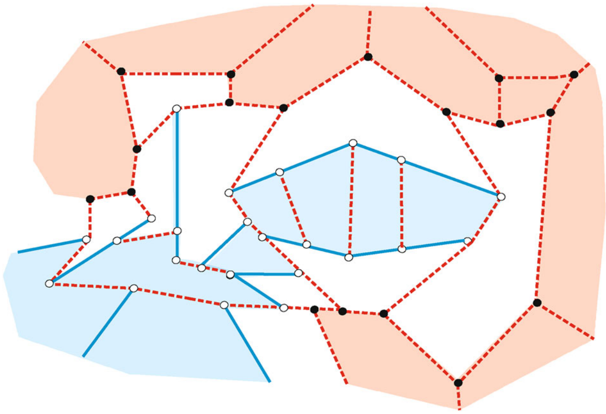

FIGURE 11 An elliptic region (*pink*), a hyperbolic region (*blue*), and the separating belt of cuspidal faces
(color figure online)

Lemma 2. *Let (H, Σ) be a generic virtual polytope, and let C be a hyperbolic region of Σ. Then*

$$Q_-^C - Q_+^C = 2\chi(C).$$

Here Q_-^C and Q_+^C are the numbers of negative and positive swallowtails adjacent to C, and $\chi(C)$ is the Euler characteristic of C.

Proof of the lemma.

Consider the region C and all adjacent to C cuspidal faces and swallowtails. Taken together, they form a cell complex Σ_C. Let F, F_{cusp}, V, V_-, and E be the number of faces, the number of cuspidal faces, the number of vertices, the number of white vertices, and the number of edges of Σ_C respectively.

Without loss of generality we can assume that $\chi(C) = \chi(\Sigma_C)$, and that each vertex of Σ_C is trivalent.

The relation $E = 3V/2$ together with the Euler formula imply that

$$F = \chi(C) + V/2. \qquad (\star)$$

For each face of Σ_C we count the number of color changes when going along the boundary. The idea of the proof is to sum up these numbers over all the faces of Σ_C.

On the one hand, we know the contribution of each face to the number of color changes. This gives

$$\sharp(\text{ color changes}) = 4(F - F_{cusp} - Q_-^C - Q_+^C) + 2F_{cusp} + 4Q_+^C.$$

On the other hand, each white vertex contributes exactly two color changes, whereas a black vertex contributes nothing. Therefore,

$$\sharp(\text{ color changes}) = 2V_-.$$

Combining the two identities with $(*)$, we get

$$2F_{cusp} + 2Q_-^C + 2Q_+^C - 2(V - V_-) = 4\chi(C) - 2Q_-^C + 2Q_+^C. \qquad (**)$$

An important observation is that cuspidal faces and swallowtails form several belts. Indeed, start with a cuspidal face (or with a swallowtail) σ_1. It has exactly two edges with endpoints of different colors. Take one of the edges. It is incident to some other face σ_2 which is again either a cuspidal face or a swallowtail. In turn, σ_2 is incident to some (cuspidal face or a swallowtail) σ_3. Proceeding this way, we get a sequence of faces which closes sooner or later in a belt.

By this reason, the left-hand side of $(**)$ is zero, which completes the proof of the lemma.

To prove the theorem, one has to sum up the identities from Lemma 2 over all hyperbolic regions. Each swallowtail face is adjacent to exactly one hyperbolic region, so we count it exactly once. Keeping in mind that $X_+ + X_- = 2$, we get immediately

$$Q_- - Q_+ = 2\sum_{hyp} \chi(C) = 2X_- = X_- + 2 - X_+.$$

□

Corollary 1. *A hyperbolic region without adjacent swallowtails is homeomorphic to the annulus.* □

The above lemma and the corollary are true for the smooth case [5].

4. Projective hedgehogs without swallowtails

Similarly to smooth hedgehogs, a virtual polytope is called *projective* if its support function is odd, that is, if $h(x) \equiv -h(-x)$. A challenging problem is the discrete version of Problem 1:

Problem 2. *Does there exist a generic projective virtual polytope with no swallowtails?*

The fan of the projective virtual polytope is *antisymmetric*. This means that the graph Σ is symmetric with respect to the central symmetry, and the central symmetry map reverses the colors of the edges.

An intermediate problem is therefore the following one:

Problem 3. *Does there exist a properly colored (that is, with local colorings as in Fig. 3) antisymmetric embedded graph Σ with no swallowtails?*

The answer is "yes", such a graph is presented in Fig. 12, but this particular graph corresponds to no virtual polytope, as in the Maurits Escher's impossible world. Consequently, Problem 2 remains unsolved.

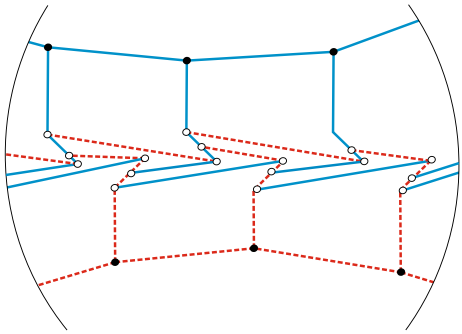

FIGURE 12 An embedded graph with no swallowtails

5. Generic virtual polytopes are everywhere dense

The genericity concept implies that generic smooth hedgehogs form an everywhere dense set among smooth hedgehogs. In this section we show that the same is true for generic simplicial virtual polytopes.

Proposition 1. *Assume that h is the support function of a simplicial virtual polytope (H, Σ). There exists a generic simplicial virtual polytope (H', Σ') whose support function h' is arbitrary close to h in $C^1(\mathbb{S}^2)$.*

Proof. The proof is based on the following *face breaking technique* (Fig. 13). Assume that σ is a face of the fan Σ. By definition, h is linear on σ. We replace the restriction of h on the face σ by a function h' which is close to h, but which is no longer linear but is a continuous patch of two linear functions. For the other faces of Σ we keep the same function h, taking a special care to maintain it continuous. This means that the vertices of the face σ get a bit shifted, and one (or several) extra edges are added inside the face σ. □

We shall gradually apply the face breaking technique to make (H, Σ) generic.

(1) A preliminary construction is the following *net*: Take the outer normal fan of the regular octahedron, or, equivalently, decompose the sphere into six polygonal regions. Each region is a combinatorial rectangle. Replace each of the regions by a grid refinement.

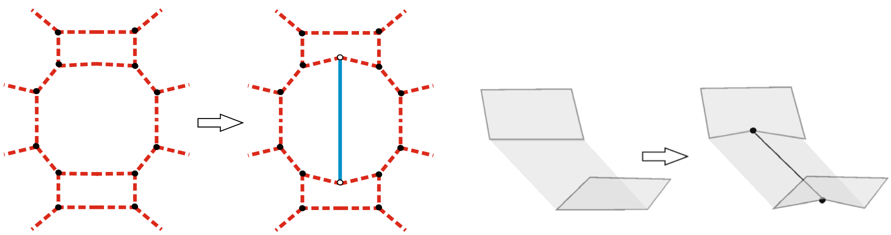

FIGURE 13 The face breaking technique

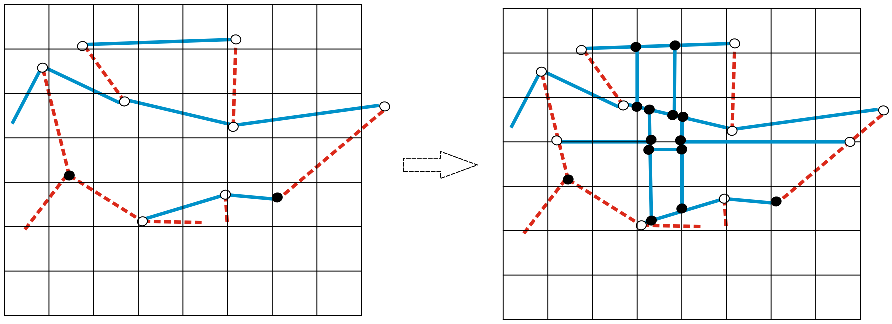

Figure 14 Step 1

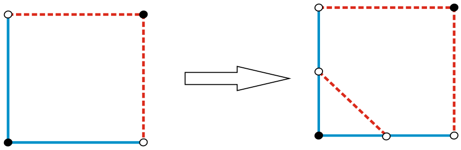

Figure 15 Steps 2 and 3

(2) We first make all the faces satisfy conditions (1) and (2) from Definition 1. For this, we cover the fan Σ with a sufficiently fine net and break all the faces along some lines of the net (see Fig. 14), keeping the graph trivalent. This yields a virtual polytope (H_1, Σ), such that the shapes of the face are "simple".

(3) Next, we make the virtual polytope to satisfy (3). Assume that the color of vertices for a face σ changes more than twice. A simple case analysis shows that it is always possible to break the face in such a way that the number of color changes at the adjacent faces remains the same, and that the face σ falls into two faces with a smaller number of vertices color changes. Figure 15 gives an example of how a face with four color changes yields two faces with two color changes each.

(4) Finally, by the same technique we force all the faces to satisfy condition (4).

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