Can hedgehogs be useful for Geometric Tomography ?

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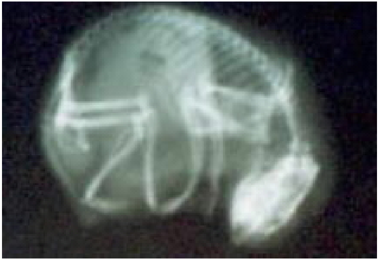

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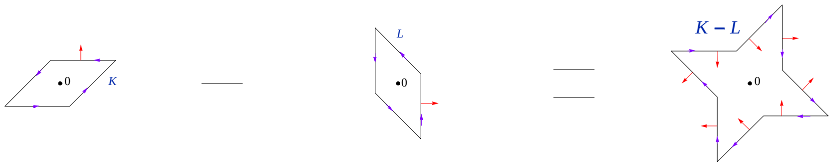
Introduction

Hedgehogs are geometrical objects that describe the **Minkowski differences of arbitrary convex bodies** in $\mathbb{R}^{n+1}$.



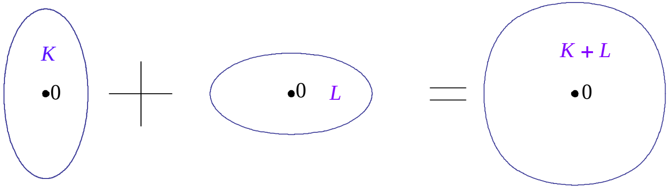
$h_K(x_1,x_2) := |x_1| + \frac{|x_1+x_2|}{\sqrt{2}}$ $\qquad$ $h_L(x_1,x_2) := \frac{|x_2-x_1|}{\sqrt{2}} + |x_2|$ $\qquad$ $h_{K-L} := h_K - h_L.$

The idea of using Minkowski differences of convex bodies may be traced back to some papers by A.D. Alexandrov and H. Geppert in the 1930's.

Many notions extend to hedgehogs and quite a number of classical results find their counterparts. Of course, a few adaptations are necessary.
In particular, volumes have to be replaced by their algebraic versions.

Minkowski sum, Minkowski difference?

- Minkowski addition: For any convex bodies K, $L \subset \mathbb{R}^{n+1}$, $K + L = \{x + y \,|\, x \in K, y \in L\}$ is still a convex body.



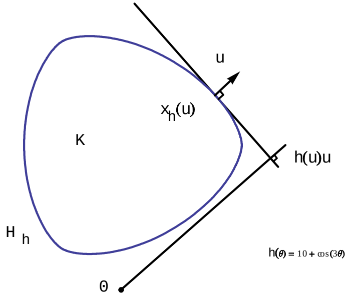
- Is there an inverse operation (a way of substracting convex bodies)?

$\hookrightarrow$ In general, there is no convex body H such that $L + H = K$.

$\hookrightarrow$ In the larger setting of « hedgehogs », which retain many properties of convex bodies, the « Minkowski difference » $M := K - L$ always exists as a geometrical object!

Support functions

Let $K \subset \mathbb{R}^{n+1}$ be a convex body (i.e., K compact, convex, $\neq \varnothing$).
K determined by its support function $h : \mathbb{S}^n \longrightarrow \mathbb{R}$, $u \longmapsto \sup_{x \in K} \langle x, u \rangle$.



If K is of class C^2_+, then ∂K is determined by $h \in C^2(\mathbb{S}^n; \mathbb{R})$ as the envelope $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ of the hyperplanes $\langle x, u \rangle = h(u)$.

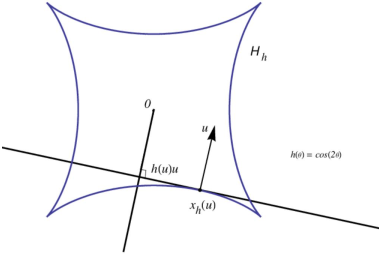
Parametrization of smooth hedgehogs

The natural parametrization of the envelope $\mathcal{H}_h$ of the hyperplanes with equation $\langle x, u\rangle = h(u)$ assigns to each $u \in \mathbb{S}^n$, the unique solution of

$$\left\{ \begin{array}{c} \langle x, u\rangle = h(u) \\ \langle x, . \rangle = dh_u(.) \end{array} \right. ,$$

that is $\boxed{x_h(u) = h(u)u + (\nabla h)(u)}$. In fact, $\mathcal{H}_h = x_h(\mathbb{S}^n)$ is defined for any $h \in C^2(\mathbb{S}^n; \mathbb{R})$. It is called **hedgehog** with support function h.

Langevin-Levitt-Rosenberg, 1988



At each regular point $x_h(u) \in \mathcal{H}_h$
u is normal to $\mathcal{H}_h$

Case of convex bodies with positive Gauss curvature

Subtracting two convex hypersurfaces (with positive Gauss curvature) by subtracting the points corresponding to a same outer unit normal to obtain a (possibly singular and self-intersecting) hypersurface:

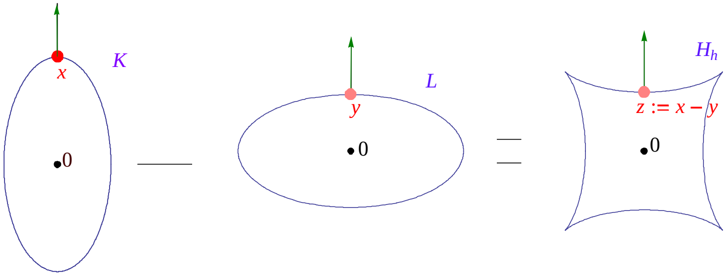
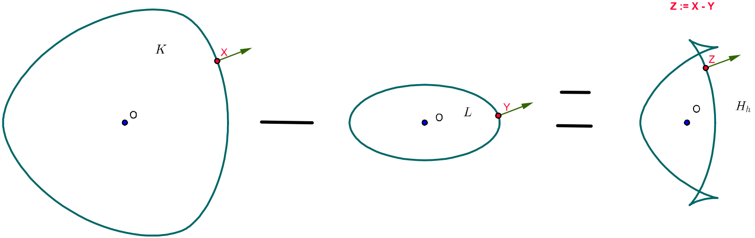


Figure: C^2-hedgehogs as differences of convex bodies of class C^2_+

$$k(\theta) := \sqrt{\cos^2\theta + 4\sin^2\theta};\quad k(\theta) := \sqrt{\cos^2\theta + 4\sin^2\theta};\quad (\theta \in [0, 2\pi]);\qquad h := k - l$$

Possibly singular and self-intersecting hypersurfaces

C^2-hedgehogs of $\mathbb{R}^{n+1}$ are such « Minkowski differences » of convex bodies of class C^2_+ of $\mathbb{R}^{n+1}$:

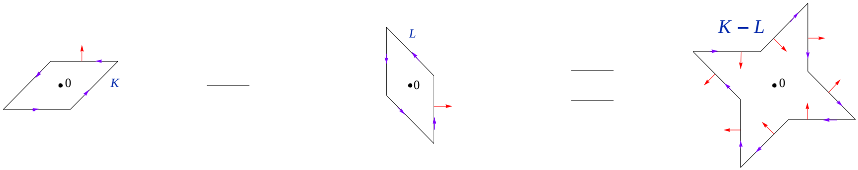


$k(\theta) := 3 - (1/4)\cos(3\theta)$ $\quad l(\theta) := \sqrt{4\cos^2\theta + \sin^2\theta}$ $\quad h := k - l.$

C^2-hedgehogs of $\mathbb{R}^{n+1}$ will constitute a real linear space which contains convex bodies of class C^2_+ of $\mathbb{R}^{n+1}$.

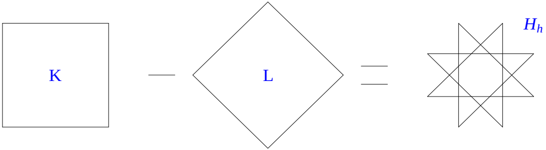
Hedgehogs as differences of arbitrary convex bodies

- Let $\left(\mathcal{K}^{n+1}, +, .\right)$ be the set of convex bodies of $\mathbb{R}^{n+1}$ equipped with Minkowski addition and multiplication by nonnegative real numbers:
  $K + L = \{x + y \,|x \in K, y \in L\}$; $\lambda.K = \{\lambda x \,|x \in K\}$.
- Not a linear space: no subtraction in $\mathcal{K}^{n+1}$.
- Formal differences $K - L$ do constitute a linear space $\left(\mathcal{H}^{n+1}, +, .\right)$.
- Any formal difference $K - L$ of two convex bodies K, $L \in \mathcal{K}^{n+1}$ has a nice geometrical representation in $\mathbb{R}^{n+1}$, (Y.M.[2], Canad. J. Math 2006).



(Proceed by induction on n, replacing support sets by 'support hedgehogs').

Hedgehogs via Euler integration



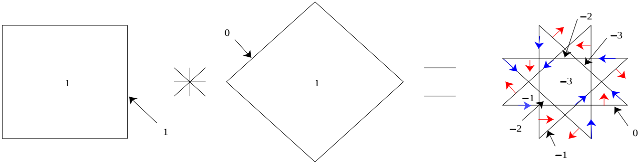
$h_K(x) = |\langle x, e_1\rangle| + |\langle x, e_2\rangle|$, $h_L(x) = |\langle x, e_3\rangle| + |\langle x, e_4\rangle|$, $(x \in \mathbb{R}^2)$;
(e_1, e_2) the canonical basis of $\mathbb{R}^2$, $e_3 = \frac{1}{\sqrt{2}}(e_1 + e_2)$ and $e_4 = \frac{1}{\sqrt{2}}(e_1 - e_2)$.

Convolution (with respect to Euler characteristic) of indexes:

$$\boxed{(-1)^{n+1}\left(\mathbf{1}_K * \mathbf{1}_{\overset{o}{-L}}\right)(x) = \mathbf{1}_h(x)}$$

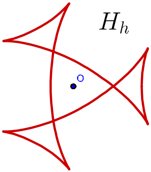
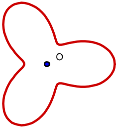
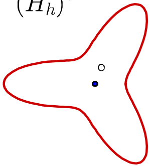
Polytopal case: Pukhlikov-Khovanskii, 1993.

Other cases: Y.M. 2 Beitr. Alg. Geom. , to appear.



Remark on duality

Star bodies are related to **hedgehogs** by **duality** (see Y.M.[2] Ann. Polon. Math. 1997):

hedgehog	pedal	dual = a star body
H_h 	$P(H_h)$ 	$(H_h)^*$ 
$u \longmapsto x_h(u)$	$u \longmapsto h(u)u$	$u \longmapsto \dfrac{u}{h(u)}$

Curvature, area, volume

- Singularities = points where the Gauss curvature $\kappa_h(u) = 1/\det[T_p x_h]$ is infinite.

- **Curvature function** $R_h := 1/\kappa_h$ well-defined and continuous on $\mathbb{S}^n$.

The Minkowski Problem arises for hedgehogs.

- (Algebraic) area measure: $s(h,\Omega) := \int_\Omega R_h(p)\, d\sigma(p)$ ($\Omega \subset \mathbb{S}^n$ Borel set).

- (Algebraic) **area**: $\boxed{s(h) := \int_{\mathbb{S}^n} R_h(p)\, d\sigma(p) = s_+(h) - s_-(h)}$, where $s_+(h)$

(resp. $s_-(h)$) is the total area of the regions of $\mathcal{H}_h$ on which $\kappa_h > 0$ (resp. < 0).

- (Algebraic) **volume**: $\boxed{v_{n+1}(h) := \int_{\mathbb{R}^{n+1}} i_h(x)\, dx = \frac{1}{n+1} \int_{\mathbb{S}^n} h(u) R_h(u)\, d\sigma(u)}$,

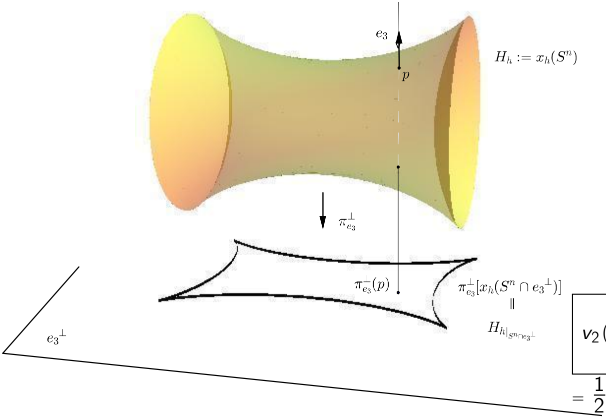
where

$i_h(x) := \deg \left[\mathbb{S}^n \to \mathbb{S}^n, u \longmapsto \frac{x_h(u) - x}{\|x_h(u) - x\|} \right]$ = degree of x with respect to $\mathcal{H}_h$.

Hedgehogs behave well under projections

Example in $\mathbb{R}^3$:

$h(x,y,z):=1+4z-3x^2, \big((x,y,z)\in\mathbb{S}^2\subset\mathbb{R}^3\big)$; $\quad h_{e_3^\perp}(x,y):=1-3x^2, \big((x,y)\in\mathbb{S}^1\subset\mathbb{R}^2=e_3^\perp\big)$, $e_3:=(0,0,1)$.



(Y.M.[2] Publ. Mat.2001)

$$\boxed{i_{h_{e_3^\perp}}(q)=\tfrac{1}{2}\left(n_h^+(q)-n_h^-(q)\right)}$$

$q:=\pi_{e_3^\perp}(p)$

$$\boxed{v_2(h_{e_3^\perp}):=\frac{1}{2}\int_{\mathbb{S}^2}|\langle u,n\rangle|R_h(u)\,d\sigma(u)}$$

$=\dfrac{1}{2}$(algebraic) projected area of $\mathcal{H}_h$ onto $e_3^\perp$.

Projection hedgehog

The notion of projection body extends to hedgehogs.

Theorem (Y.M.^{2}- 2001, Adv. Math). *Let $\mathcal{H}_f$ be a hedgehog of $\mathbb{R}^{n+1}$ and let R_f be its curvature function. Then*

$$\boxed{h_f : \mathbb{S}^n \to \mathbb{R},\ p \longmapsto \frac{1}{2}\int_{\mathbb{S}^n} |\langle p, q\rangle|\, R_f\,(q)d\sigma(q)},$$

is of class C^2 and hence defines a hedgehog Π_f.

Prop.(ib.) $h_f\,(p) =$ the n-dimensional volume of $\mathcal{H}_{f_p}$, where $f_p := f_{|\mathbb{S}^n \cap p^\perp}$.

Def. Π_f *is called the* **projection hedgehog** *of $\mathcal{H}_f$.*

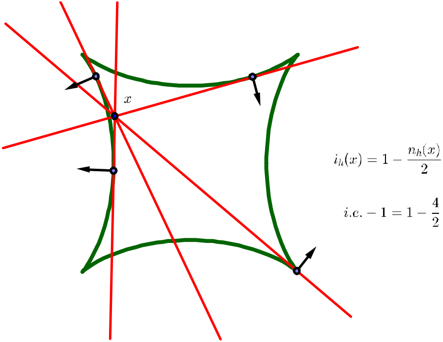
If f is the restriction to $\mathbb{S}^n$ of a sublinear function on $\mathbb{R}^{n+1}$, then $\mathcal{H}_f$ is the boundary of a convex body K and Π_f is the **projection body** of K.

Remark on the index in the planar case

The index $i_h(x)$ of x with respect to $\mathcal{H}_h \subset \mathbb{R}^2$ can be evaluated by counting the number of cooriented support lines through x :

$$\boxed{i_h(x) = 1 - \tfrac{1}{2} n_h(x)},$$

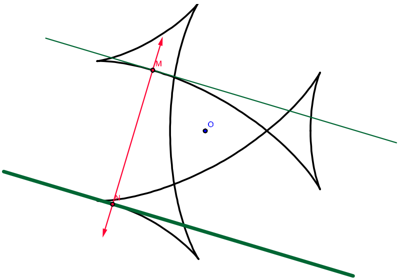
where $n_h(x)$ = the number of cooriented support lines of $\mathcal{H}_h$ through x (Y.M.[2] Publ. Mat.2001, Canad. J. Math. 2006)



Constant width

The notion of being of constant width extends to hedgehogs.

$\mathcal{H}_h \subset \mathbb{R}^{n+1}$ is said to be of **constant width** if the distance $h(u)+h(-u)$ between the two support hyperplanes orthogonal to $u \in \mathbb{S}^n$ does not depend on u.

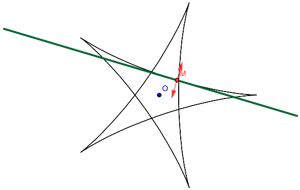


A hedgehog of constant width

Projective hedgehogs

Hedgehogs of constant width 0 are said to be **projective**:

$$(\mathcal{H}_h \text{ projective}) \Longleftrightarrow (\forall u \in \mathbb{S}^n, \; h(-u) = -h(u)).$$



A projective hedgehog

They are parametrized by the projective space $\mathbb{R}P^n := \mathbb{S}^n/\text{antipody}$.

Decomposition of convex bodies

Study of convex bodies (or hypersurfaces) by decomposition.

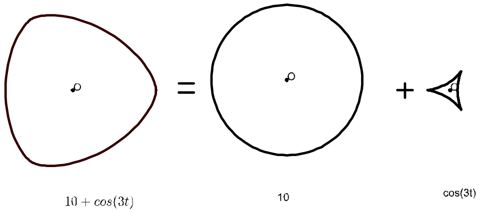
Examples:

- *Conjectured characterization of the 2-sphere* (A.D. Alexandrov, mid 1930's).

$\hookrightarrow$ Write $S = S\left(0_{\mathbb{R}^3}; r\right) + \left(S - S\left(0_{\mathbb{R}^3}; r\right)\right)$ and study $\left(S - S\left(0_{\mathbb{R}^3}; r\right)\right)$ using orthogonal projections techniques $\left(\mathrm{Y.M.}^2 \text{ C. R. Acad. Sci. Paris 2001}\right)$.

- *Study of convex bodies of constant width 2r.*

$\hookrightarrow$ Write $K =$ (A sphere of radius r) $+$ (A projective hedgehog) and study the latter.



Geometrization of analytical problems

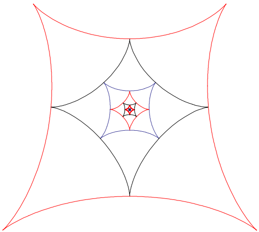
Geometrization of problems by considering functions as support functions.

Example. **Geometrical proof/interpretation of the Sturm-Hurwitz theorem** (Y.M.[2], Arch. Math. 2003): *Any continuous 2π-periodic real function*

$$h(\theta) = \sum_{n=N}^{+\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

has at least as many zeros as its first nonvanishing harmonics:

$$\boxed{\#\{\theta \in [0, 2\pi[\,|\, h(\theta) = 0\} \geq 2N}.$$



The Minkowski problem can be extended to hedgehogs

• **The classical Minkowski problem**:

Existence, uniqueness and regularity of a closed convex hypersurface of $\mathbb{R}^{n+1}$ whose Gauss curvature is prescribed as a positive function on $\mathbb{S}^n$.

• **Important role in:**

- the theory of convex bodies.
- the theory of elliptic Monge-Ampère equations.

• **Its natural extension to hedgehogs**:

- **A way for exploring Monge-Ampère equations of mixed type.**
- A calculation gives: $R_h(u) = \det\left[H_{ij}(u) + h(u)\,\delta_{ij}\right]$, where $\left(H_{ij}(u)\right)$ is the Hessian of h at u with respect to an orthonormal frame on $\mathbb{S}^n$.

Case n = 2

In dimension $n+1>2$, the problem is very difficult. We will only consider the case $n=2$.

• The curvature function of $\mathcal{H}_h \subset \mathbb{R}^3$ is given by

$$1/\kappa_h = h^2 + h\Delta_2 h + \Delta_{22} h$$

(Δ_2 is the Laplacian and Δ_{22} the Monge-Ampère operator, i.e. the sum and the product of the eigenvalues of *Hess* h).

• The type of the equation $h^2 + h\Delta_2 h + \Delta_{22} h = 1/\kappa$ is given by $sgn\,[1/\kappa]$. So, the PB leads to **PDE's of mixed type for non-convex hedgehogs**.

Key classical results

- Major contributions by Minkowski, Alexandrov, Nirenberg, Pogorelov, Cheng-Yau and others.
- Existence of a weak solution:

Theorem (Minkowski - 1903)

If $\kappa \in C(\mathbb{S}^n;\mathbb{R})$ is positive and such that

$$\int_{\mathbb{S}^n} \frac{u}{\kappa(u)}\, d\sigma(u) = 0$$

then κ is the Gauss curvature of a unique (up to translation) closed convex hypersurface $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$.

- Strong result:

Theorem (Pogorelov - 1975, Cheng and Yau - 1976)

If $\kappa \in C^m(\mathbb{S}^n;\mathbb{R})$, with $m \geq 3$, then: $\forall \alpha \in \left]0,1\right[$, $h \in C^{m+1,\alpha}(\mathbb{S}^n;\mathbb{R})$.

Existence problem

Existence of a $\mathbf{C}^2$-solution

What are necessary and sufficient conditions for $R \in C(\mathbb{S}^n; \mathbb{R})$ to be the curvature function of some hedgehog $\mathcal{H} = \mathcal{K} - \mathcal{L}$?

- (1) $\int_{\mathbb{S}^n} uR(u)\, d\sigma(u) = 0$ still necessary (but not sufficient: consider -1).

- Equations with no solution (Y.M.[2], Adv. in Math. 2001):

For every $v \in S^2$, $R(u) = 1 - 2\langle u, v\rangle^2$ satisfies (1) and changes sign cleanly on S^2 but is not a curvature function:

there is no $h \in C^2(\mathbb{S}^2;\mathbb{R})$ such that $R_h = R$.

- Can the curvature function of a hedgehog $\mathcal{H}_h$ be nonpositive on $\mathbb{S}^2$?

Hedgehog with everywhere nonpositive curvature function

Conjecture (C): *If $S \subset \mathbb{R}^3$ is a closed convex surface of class C^2_+ such that*

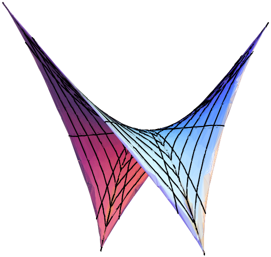
$$(k_1 - c)(k_2 - c) \leq 0,$$

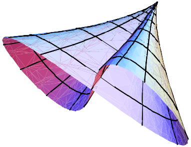
with $c = cst$, then S must be a sphere of radius $1/c$.

(C) is equivalent to (H):

(H) *If $\mathcal{H}_h \subset \mathbb{R}^3$ is a hedgehog such that $R_h \leq 0$, then $\mathcal{H}_h$ is a point.*

Counter-example to (H) (Y.M.[2], C. R. Acad. Sci. Paris 2001).





Uniqueness problem

Uniqueness of a C^2-solution:

Let $R \in C(\mathbb{S}^n;\mathbb{R})$ be the curvature function of some hedgehog $\mathcal{H}_h$. What are necessary and sufficient conditions on R for $\mathcal{H}_h$ to be uniquely determined by R (up to parallel translations and identifying h with $-h$)*?*

In the convex case, the uniqueness comes from the equality condition in a well-known Minkowski's inequality. This inequality cannot be extended to hedgehogs and uniqueness is lost.

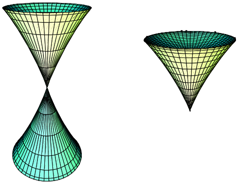


Figure: Noncongruent smooth (but not analytic) hedgehogs with the same curvature function

Question. *Does there exist any pair of noncongruent analytic hedgehogs with the same curvature function?*

Results relative to the uniqueness

Let H_3 be the linear space of C^2-hedgehogs defined up to a translation in $\mathbb{R}^3$.

Theorem (Y.M.[2], Central European J. Math. 2012). *Let $\mathcal{H}$ and $\mathcal{H}'$ be C^2-hedgehogs that are linearly independent in* H_3. *If some linear combination of $\mathcal{H}$ and $\mathcal{H}'$ is of class C^2_+, then $\mathcal{H}$ and $\mathcal{H}'$ have distinct curvature functions.*

Our second result relies on the extension to hedgehogs of the notion of mixed curvature function.

Theorem (Y.M.[2], Central European J. Math. 2012). *Let $\mathcal{H}$ and $\mathcal{H}'$ be analytic (resp. projective C^2) hedgehogs of $\mathbb{R}^3$ that are linearly independent in* H_3.
If the mixed curvature function of $\mathcal{H}$ and $\mathcal{H}'$ does not change sign, then $\mathcal{H}$ and $\mathcal{H}'$ have distinct curvature functions.

Example of a uniqueness result

The following result relies on the decomposition of hedgehogs into centered and projective parts.

Theorem (Y.M.[2], Central European J. Math. 2012). *Let $\mathcal{H}$ and $\mathcal{H}'$ be C^2-hedgehogs that are linearly independent in* H_3 *and the centered parts of which are non-trivial and proportional to one and the same convex surface of class C^2_+. Then $\mathcal{H}$ and $\mathcal{H}'$ have distinct curvature functions.*

Corollary. *Two C^2-hedgehogs of nonzero constant width that are linearly independent in* H_3 *must have distinct curvature function.*

Consequence. The Monge-Ampère equation $h^2 + h\Delta_2 h + \Delta_{22} h = R$, $R \in C\left(\mathbb{S}^2;\mathbb{R}\right)$, cannot admit more than one solution of the form $f + r$, where $f \in C^2\left(\mathbb{S}^2;\mathbb{R}\right)$ is antisymmetric and r is a nonzero constant.

(Solutions are identified if they are opposite or if they differ by the restriction to $\mathbb{S}^2$ of a linear form on $\mathbb{R}^3$)

Thank you very much for your attention!

Figure: European hedgehog